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The Interaction of γ-Rays with Mesotrons

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Differential cross sections for the production of a mesotron pair by a γ-ray and for the bremsstrahlung of a mesotron in the electromagnetic field of a nucleus have been calculated. Both the Proca wave field and the Kemmer matrix formulations of the theory for mesotrons of unit spin and unit magnetic moment were used. These differential cross sections have been integrated in the limit where the energies of the mesotrons and photon are large compared to the rest energy of the mesotron. For a pure Coulomb field, the integrated cross sections are $A\alpha Z^2e^4E^2/(\mu c^2)^4$ where for pair production $A=19/72$ and E is the γ-ray energy, and for bremsstrahlung $A=11/72$ and E is the initial mesotron energy. Since in these processes the important impacts are much closer than the nuclear radius, these cross sections do not describe correctly the electromagnetic effects in the neighborhood of an actual nucleus. The cross sections in a field of the form $(Ze/r)(1-e^{-r/d})$ where d is the nuclear radius and is taken to be $5\hbar Z^{1/3}/6\mu c$, are $B'\alpha Z^{5/3}e^4E/(\mu c^2)^3$ where E is defined as before and $B'=\pi/6$ and $\pi/18$ for pair production and bremsstrahlung, respectively. Since considerations of the validity of the Born approximation method used show that the leading terms in the cross sections lose their validity before they become dominant, terms of lower order which give cross sections increasing only logarithmically with the energy have been calculated by the method of virtual quanta. Further, to estimate the minimum values of the cross sections beyond the range of validity, the frequency integral in the virtual quantum calculation was cut off at $k/\mu c^2=\hbar c/e^2$.

I

THE absorption and production of the penetrating component of cosmic rays depend, among other processes, on the bremsstrahlung and pair production of mesotrons in matter. These processes are also involved in any account of the secondary soft radiation associated with the penetrating component under great thicknesses of matter. Oppenheimer, Snyder and Serber[1] have estimated that in a Coulomb field these cross sections are $\sim\alpha Z^2e^4E^2/(\mu c^2)^4$; whereas in an actual nuclear electric field, which they take to be a Coulomb field cut off at the nuclear radius $d\sim\hbar Z^{\frac{1}{3}}/\mu c$, the cross sections are $\sim\alpha Z^{5/3}e^4E/(\mu c^2)^3$. The large difference between these two forms is due to the high frequency Fourier components of the Coulomb field which are missing in the cut-off field. Kobayasi and Utiyama[2] have obtained similar expressions for the cross sections in a cut-off Coulomb field by the approximate method of virtual quanta. Because the approximations involved in this method lose their validity for just those small impact parameters and high frequencies which are most important, these results are still uncertain by a multiplicative factor of order unity.

[1] J. R. Oppenheimer, H. Snyder and R. Serber, Phys. Rev. **57**, 75 (1940).

[2] M. Kobayasi and R. Utiyama, Sci. Pap. Inst. Phys. Chem. Research, Tokyo **37**, 221 (1940). *Note added in proof.*—Since our manuscript was sent to press, a paper by M. Kobayasi and R. Utiyama, Proc. Phys. Math. Soc. Japan **22**, 882 (1940), has appeared in which bremsstrahlung and pair production cross sections at high energies are calculated by eliminating high frequencies, $k>\hbar c/e^2$, as in this paper.

Booth and Wilson[3] have used the method of virtual quanta to obtain a cross section for bremsstrahlung which, however, is incorrect because they fail to distinguish between a Coulomb field and the actual nuclear electric field near the nuclear radius. Their result does not agree with the cross section in a pure Coulomb field because the method fails completely in that application. In order to refine the conclusions of Oppenheimer, Snyder and Serber, we have calculated by Born approximation, in the limit where the energies are large compared to the rest energy of the mesotron, those terms in the cross sections for bremsstrahlung and pair production of mesotrons which increase most rapidly with the energy. We have in addition, calculated further terms in the cross section for bremsstrahlung by the method of virtual quanta.

The mesotron of unit spin is usually described by Proca's equation which assigns also unit magnetic moment. Corben and Schwinger[4] point out that a consistent theory can be developed for particles of unit spin and arbitrary magnetic moment, but conclude from cosmic-ray data that of these the unit magnetic moment is most likely. In the present paper, also, the theory describing a mesotron of gyromagnetic ratio unity is used since it gives a unique cancellation in the matrix elements of the current and thus leads to cross sections increasing least rapidly with the energy.

Both the Proca wave field and the Kemmer[5] matrix formulation of the theory were used to calculate the differential cross sections. The first method is considerably longer and more tedious but yields the added information that at high energies it is the longitudinal-transverse mesotron spin transitions which are important. The second method requires the use of spin sum and spur techniques which, as their application to mesotron theory is new, will be described in some detail in Section II.

The calculation by Born approximation of the leading terms in the cross sections for bremsstrahlung and pair creation gives the following results. The cross sections in the Coulomb field Ze/r are

$$\sigma = A\alpha Z^2 e^4 E^2/(\mu c^2)^4, \tag{1}$$

where for bremsstrahlung, $A=11/72$ and E is the initial mesotron energy; and for pair creation, $A=19/72$ and E is the γ-ray energy. In complete contrast to the Dirac theory in this high energy limit, these impacts involve large angles of scattering and radiation with large momentum transfers of the order E/c. These are associated with high frequency Fourier components of the Coulomb field which are present only at distances $\sim\hbar c/E$, which, for $E \gg \mu c^2$, is much less than the actual nuclear radius where the electric field is certainly not Coulombian. Thus, in order to obtain cross sections with any significance, it is necessary to consider a more exact approximation to the electric field near the nucleus. The atomic screening can be neglected since only close impacts contribute appreciably.

In connection with this treatment of the electric field near the nucleus, it is, of course, important to remember that all specifically nuclear reactions and couplings are being neglected. In fact we have at present no method at all adequate for handling the large couplings presumably involved. The electromagnetic effects we treat are thus not a complete description of the collisions, but constitute a sort of minimum prediction largely independent of the specifically nuclear effects, characterized by different angular distributions, different reaction products and different ranges of the impact parameter.

A closer approximation to the nuclear potential is $(Ze/r)(1-e^{-r/d})$, where d is the nuclear radius and is taken to be $5\hbar Z^{\frac{1}{3}}/6\mu c = 1.82\times 10^{-13} Z^{\frac{1}{3}}$ cm for $\mu=177$ electron masses. This essentially sets an upper limit of about $\mu c/Z^{\frac{1}{3}}$ on the momentum transfer to the nucleus and gives

$$\sigma = \frac{B\alpha Z^2 e^4 E^2}{(\mu c^2)^4}\frac{\hbar c}{Ed} = \frac{B'\alpha Z^{5/3} e^4 E}{(\mu c^2)^3}, \tag{2}$$

where E has the two meanings stated above, and $B'=\pi/18$ and $\pi/6$, respectively, for bremsstrahlung and pair creation. In contrast to the virtual quanta calculation of these cross sections which cannot take into account the shape of the potential near the nucleus, the constant B can be determined definitely for any assumed shape.

[3] F. Booth and A. H. Wilson, Proc. Roy. Soc. **A175**, 483 (1940).
[4] H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1940).
[5] N. Kemmer, Proc. Roy. Soc. **A173**, 91 (1939).

Since these cross sections are derived by perturbation methods, their validity is limited to energies where the perturbing interactions with the Coulomb field and with the radiation field are small. Oppenheimer, Snyder and Serber[1] show that the more incisive limitation arises from the requirement that the coupling with the radiation field be small which sets an upper limit $\sim 10^{10}$ ev on the validity of the formulae. Under these circumstances, where the leading terms in the cross sections lose their validity before they become dominant, it can be of interest to investigate terms of lower order, which give cross sections increasing only logarithmically with the energy. The impacts responsible for these terms are relatively distant and the method of virtual quanta is thus applicable (see Section IV). One of these terms shows the "infra-red catastrophe," so that their order of magnitude is most conveniently demonstrated by the cross section for fractional energy loss of a mesotron by bremsstrahlung

$$\sigma=\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2\left[\frac{5\pi}{144}\frac{E}{\mu c^2 Z^{\frac{1}{3}}}+\frac{7}{72}\ln^3\frac{2\pi E}{5\mu Z^{\frac{1}{3}}}-\frac{23}{96}\ln^2\frac{2\pi E}{5\mu Z^{\frac{1}{3}}}+\left(\frac{7\pi^2}{72}+\frac{121}{48}\right)\ln\frac{2\pi E}{5\mu Z^{\frac{1}{3}}}\right]. \quad (3)$$

The first term in this expression is determined by the cross section (2). The other terms must be regarded as having the characteristic uncertainty of the method of virtual quanta: they contain undetermined factors of order unity in the arguments of the logarithms. The last term is, in structure and origin, similar to the dominant term in the corresponding cross section for bremsstrahlung of particles of half integral spin and unit magnetic moment, e.g., electrons. The impacts responsible for it are relatively distant; the couplings are always small, being of the order $(e^2/\hbar c)^{\frac{1}{2}}$. The terms in $\ln^3 E$ and $\ln^2 E$ result from two features not present in calculations based on the Dirac theory. First is the existence of terms in the cross section which depend on the impact parameter as $\ln^2 r$ which weights close impacts and high energies more than for the Dirac particle where the dependence is as $\ln r$. Second is the presence of terms in $1/(1-\epsilon)$, $0\leqslant\epsilon\leqslant 1-(\mu c^2/E)$, where ϵ is the fractional energy transfer to the γ-ray, which weight large energy transfers. The term in $\ln^2 r$ depends on the frequency of the virtual quanta as $\ln^2 k$ and at high energies, involve high frequencies and correspondingly large couplings. Thus the arguments of Oppenheimer, Snyder and Serber, which limit the validity of the first term proportional to E to frequencies of the order $(\hbar c/e^2)^{\frac{1}{2}}$ in the symmetric coordinate system, or to $2\hbar c/e^2$ in the mesotron rest system in which the virtual quantum calculation is performed, must also be invoked to limit the validity of the $\ln^3 E$ and the $\ln^2 E$ terms arising from this $\ln^2 k$ term in the cross section.

With the leading terms in (3) so limited in validity, it is of interest to calculate at least the lower limit for their contribution to the cross section at energies $>10^{10}$ ev. This is accomplished by making a virtual quantum calculation of all terms in the cross section and cutting off the frequency integral at $k/\mu c^2=A\leqslant 2\hbar c/e^2$. The cutoff A cannot be assigned exactly but must certainly be greater than 10 if relativistic mesotron theory and the effects due to spin are to have any meaning; there seems to be no *a priori* theoretical reason why A should not be of order 100. Calculations were made with $A=\hbar c/e^2=137$. The Born approximation calculations of (1) and (2) are discussed in Section III. The virtual quantum calculations and their modifications of (3) are given in Section IV. The application of these calculations to a discussion of cosmic-ray bursts appears in a separate paper.

II

Kemmer's wave equation for the mesotron of mass μ, charge $\pm e$, spin 1 and magnetic moment 1 is

$$\partial_\rho\beta_\rho\psi+K\psi=0, \quad (4)$$

where $K=\mu c/\hbar$, $\partial_\rho=\partial/\partial x_\rho$, $x_4=ict$ and the operators β_ρ are ten-row square matrices obeying the commutation rules

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\beta_\mu\delta_{\nu\rho}+\beta_\rho\delta_{\nu\mu}, \quad (5)$$

the charge and the energy are given by

$$ne=e\int\psi^*\beta_4\psi d\tau;\quad E=\int\psi^*\beta_4 H\psi d\tau, \quad (6)$$

where H is the Hamiltonian defined by

$$H\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}. \tag{7}$$

From (4) Kemmer obtains the Hamiltonian

$$H = (\hbar c/i)\partial_k i(\beta_4\beta_k - \beta_k\beta_4) + \mu c^2\beta_4 \tag{8}$$

together with the supplementary condition

$$\partial_k\beta_k\beta_4{}^2\psi + (1-\beta_4{}^2)K\psi = 0. \tag{9}$$

For plane wave solutions $u(p, t)$

$$H^3u = E^2Hu,$$

so that H has the eigenvalues 0 and $\pm E$, but there are no solutions of $Hu_0 = 0$ compatible with the condition (9). The six solutions of $Hu_+ = Eu_+$ and $Hu_- = -Eu_-$ are normalized to unit charge by setting $u_+{}^*\beta_4u_+ = 1$ and $u_-{}^*\beta_4u_- = -1$. To these are adjoined, for mathematical convenience in spur methods, four solutions u_0 of $Hu_0 = 0$ in order to complete the set of ten vectors u to form a matrix which is, apart from a normalization constant, unitary. Kemmer shows that $E = \mu c^2u^*u$ which determines the normalization in agreement with unit charge.

The second quantization is carried through in the usual manner by expanding u in terms of the six solutions for positive and negative charge.

The Hamiltonian (8) is not suitable for the treatment of the interaction with the electromagnetic field since the energy matrix β_4H is not in Hermitian form and this leads to complications in the treatment of the zero states due to the perturbation of the supplementary condition. A satisfactory Hamiltonian can be obtained by using the wave equation (4) and the supplementary condition (9) explicitly in the expression (6) for the energy. There results[6]

$$H = (1/\mu)\int \psi^*\beta_4{}^2[\mu^2c^2 - \hbar^2(\partial_k\beta_k)(\partial_l\beta_l)]\beta_4{}^2\psi d\tau. \tag{10}$$

The interaction with the electromagnetic field is introduced by the usual substitution $\partial_\mu \to \partial_\mu{}^- = \partial_\mu - (ie/\hbar c)A_\mu$, $A_4 = iV$. The perturbation due to the scalar potential is then

$$H_1 = \int \psi^*\beta_4V\psi d\tau, \tag{11}$$

that linear in the vector potential is

$$H_2 = (ie\hbar/\mu c)\int \psi^*\beta_4{}^2[(A_k\beta_k)(\partial_l\beta_l) + (\partial_k\beta_k)(A_l\beta_l)]\beta_4{}^2\psi d\tau \tag{12}$$

and the term quadratic in A is

$$H_3 = (e^2/\mu c^2)\int \psi^*\beta_4{}^2(A_k\beta_k)(A_l\beta_l)\beta_4{}^2\psi d\tau. \tag{13}$$

In momentum space

$$\begin{aligned} H_1 = \int d\mathbf{k}\int d\mathbf{l}\int d\mathbf{x}\, &\exp[(i/\hbar)(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}]V(x)\sum_{i,j=1}^{3}[a_j{}^*(l)a_i(k)\exp[(i/\hbar)(E_l - E_k)t]u_+{}^{*j}(l)\beta_4u_+{}^i(k) \\ &+ a_j{}^*(l)b_i{}^*(k)\exp[(i/\hbar)(E_l+E_k)t]u_+{}^{*j}(l)\beta_4u_-{}^i(k) \\ &+ b_j(l)a_i(k)\exp[-(i/\hbar)(E_l+E_k)t]u_-{}^{*j}(l)\beta_4u_+{}^i(k) \\ &+ b_j(l)b_i{}^*(k)\exp[-(i/\hbar)(E_l+E_k)t]u_-{}^{*j}(l)\beta_4u_-{}^i(k)]. \end{aligned} \tag{14}$$

The summation over the spin indices is suppressed by writing u^* as a one-row and u as a one-column matrix. a^* creates, while a destroys, a positive mesotron, and b^* creates, while b destroys, a negative mesotron.

[6] We are grateful to Dr. Kemmer for informing us of this result by private communication.

Matrix elements of H_2 for absorption of a quantum of momentum $\mathbf{n}=\mathbf{l}-\mathbf{k}$ and polarization $\mathbf{e}$ are

$$H_2=-\frac{e}{\mu c^2}\left(\frac{2\pi\hbar^2c^2}{n}\right)^{\frac{1}{2}}\int d\mathbf{k}\int d\mathbf{l}\sum_{i,j=1}^{3}\left[a_j^*(l)a_i(k)\exp\left[(i/\hbar)(E_l-E_k)t\right]u_+^{*j}(l)\beta_4^2(k\beta_e\beta_k+l\beta_l\beta_e)\beta_4^2u_-^i(k)\right.$$
$$+a_j^*(l)b_i^*(k)\exp\left[(i/\hbar)(E_l+E_k)t\right]u_+^{*j}(l)\beta_4^2(k\beta_e\beta_k+l\beta_l\beta_e)\beta_4^2u_-^i(k)$$
$$+b_j(l)a_i(k)\exp\left[-(i/\hbar)(E_l+E_k)t\right]u_-^{*j}(l)\beta_4^2(k\beta_e\beta_k+l\beta_l\beta_e)\beta_4^2u_+^i(k)$$
$$\left.+b_j(l)b_i^*(k)\exp\left[-(i/\hbar)(E_l-E_k)t\right]u_-^{*j}(l)\beta_4^2(k\beta_e\beta_k+l\beta_l\beta_e)\beta_4^2u_-^i(k)\right],\quad(15)$$

where $\beta_p=p_k\beta_k/p$. The matrix elements of H_2 for emission of a quantum are obtained from the above by setting $\mathbf{n}=\mathbf{k}-\mathbf{l}$.

H_3 has matrix elements only for two quanta processes. They are similar to those given above with a different normalization and with $\beta_4^2(\beta_{e1}\beta_{e2}+\beta_{e2}\beta_{e1})\beta_4^2$ as the matrix operator where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors in the directions of polarization of the two quanta. In double absorption $\mathbf{n}_1+\mathbf{n}_2=\mathbf{l}-\mathbf{k}$, for double emission $\mathbf{n}_1+\mathbf{n}_2=\mathbf{k}-\mathbf{l}$, for absorption of 1 and emission of 2 $\mathbf{n}_1-\mathbf{n}_2=\mathbf{l}-\mathbf{k}$, and for emission of 1 and absorption of 2 $\mathbf{n}_1-\mathbf{n}_2=\mathbf{k}-\mathbf{l}$. H_3 is not involved in either bremsstrahlung or pair production since these appear as one quantum process in the nuclear rest system.

Let us now refer, in particular, to the creation of a positive mesotron $\mathbf{k}$ and a negative mesotron $\mathbf{l}$ (its actual momentum is $-\mathbf{l}$) by a γ-ray of momentum $\mathbf{n}$ near a nucleus of charge Ze. The conservation of energy gives

$$n=E_k+E_l=(\mu^2c^4+k^2)^{\frac{1}{2}}+(\mu^2c^4+l^2)^{\frac{1}{2}}$$

for time proportional transitions. The transition takes place by means of H_1 and H_2 through an intermediate state; H_2 conserving momentum and H_1 transferring an amount of momentum $\mathbf{q}=\mathbf{n}+\mathbf{l}-\mathbf{k}$ to the nucleus. Indicating the quantum, positive mesotron and negative mesotron in order, we have the following sets of state A: $\mathbf{n}$, $-$, $-$; intermediate state I: $-$, $\mathbf{k}$, $\mathbf{k}-\mathbf{n}$; II: $\mathbf{n}$, $\mathbf{k}-\mathbf{n}$, $\mathbf{l}$; III: $-$, $\mathbf{n}+\mathbf{l}$, $\mathbf{l}$: IV: $\mathbf{n}$, $\mathbf{k}$, $\mathbf{n}+\mathbf{l}$; and finally, F: $-$, $\mathbf{k}$, $\mathbf{l}$. The matrix element $H(FA)$ between the initial and the final states is then given by

$$H(FA)=\sum\left[\frac{H_1(F\mathrm{I})H_2(\mathrm{I}A)}{E_A-E_{\mathrm{I}}}+\frac{H_2(F\mathrm{II})H_1(\mathrm{II}A)}{E_A-E_{\mathrm{II}}}+\frac{H_1(F\mathrm{III})H_2(\mathrm{III}A)}{E_A-E_{\mathrm{III}}}+\frac{H_2(F\mathrm{IV})H_1(\mathrm{IV}A)}{E_A-E_{\mathrm{IV}}}\right],$$

where the summation is to be taken over the polarizations of the intermediate states. On inserting the explicit values on the right, we can make some reduction by combining terms and using the relations

$$\sum_{i=1}^{3}\left[u_-^i(p)u_-^{*i}(p)-u_+^i(p)u_+^{*i}(p)\right]=-\frac{H_p}{E_p}\sum_{m=1}^{10}u^m(p)u^{*m}(p)=-\frac{H_p}{\mu c^2}$$

and

$$\sum_{i=1}^{3}\left[u_+^i(p)u_+^{*i}(p)+u_-^i(p)u_-^{*i}(p)\right]=\frac{H_p^2}{E_p^2}\sum_{m=1}^{10}u^m(p)u^{*m}(p)=\frac{H_p^2}{\mu c^2E_p},$$

where m goes over $+$, 0, $-$ states and

$$H_p=\mu c^2\beta_4+ip(\beta_4\beta_p-\beta_p\beta_4).$$

The final result is

$$H(FA)=-\frac{e}{(\mu c^2)^2}\left(\frac{2\pi\hbar^2c^2}{n}\right)^{\frac{1}{2}}V(q)\left\{\frac{u_+^{*j}(k)\beta_4^2(a\beta_e\beta_a+k\beta_k\beta_e)\beta_4^2H_a(H_a-E_l)\beta_4u_-^i(l)}{E_l^2-E_a^2}\right.$$
$$\left.+\frac{u_+^{*j}(k)\beta_4H_b(H_b+E_k)\beta_4^2(l\beta_e\beta_l+b\beta_b\beta_e)\beta_4^2u_-^i(l)}{E_k^2-E_b^2}\right\},\quad(16)$$

where $V(q)$ is the Fourier component of V and $\mathbf{a}=\mathbf{k}-\mathbf{n}$, $\mathbf{b}=\mathbf{n}+\mathbf{l}$. The differential cross section for pair production is

$$d\varphi=\frac{\pi}{\hbar c}\frac{kE_klE_ld\Omega_kd\Omega_ldE_k}{(2\pi\hbar c)^6}\sum_e\sum_{i,j=1}^{3}|H(FA)|^2.$$

To perform the summations over the spins of positive and negative mesotron, we introduce the annihilation operators $\lambda_+(p)$ and $\lambda_-(p)$ such that $\lambda_+u_+=u_+$, $\lambda_+u_0=0$, $\lambda_+u_-=0$ and $\lambda_-u_+=0$, $\lambda_-u_0=0$, $\lambda_-u_-=u_-$. They are given by

$$\lambda_+(p)=(H_p/2E_p{}^2)(H_p+E_p),\quad \lambda_-(p)=(H_p/2E_p{}^2)(H_p-E_p).$$

This gives

$$d\varphi=\frac{e^2}{\hbar c}\frac{(V(q))^2}{8n(\mu c^2)^6}\frac{kld\Omega_kd\Omega_ldE_k}{(2\pi\hbar c)^4}\sum_e \mathrm{Spur}\left\{\frac{P}{(E_l{}^2-E_a{}^2)^2}+\frac{2Q}{(E_l{}^2-E_a{}^2)(E_k{}^2-E_b{}^2)}+\frac{R}{(E_k{}^2-E_b{}^2)^2}\right\},$$

where

$$P=\beta_4{}^2(a\beta_e\beta_a+k\beta_k\beta_e)\beta_4{}^2H_a(H_a-E_l)\beta_4H_l(H_l-E_l)\beta_4H_a(H_a-E_l)\beta_4{}^2(k\beta_e\beta_k+a\beta_a\beta_e)\beta_4{}^2H_k(H_k+E_k),$$

$$Q=\beta_4{}^2(a\beta_e\beta_a+k\beta_k\beta_e)\beta_4{}^2H_a(H_a-E_l)\beta_4H_l(H_l-E_l)\beta_4{}^2(b\beta_e\beta_b+l\beta_l\beta_e)\beta_4{}^2H_b(H_b+E_k)\beta_4H_k(H_k+E_k),$$

$$R=\beta_4H_b(H_b+E_k)\beta_4{}^2(l\beta_e\beta_l+b\beta_b\beta_e)\beta_4{}^2H_l(H_l-E_l)\beta_4{}^2(b\beta_e\beta_b+l\beta_l\beta_e)\beta_4{}^2H_b(H_b+E_k)\beta_4H_k(H_k+E_k).$$

In evaluating these spurs, it is convenient to generalize the commutation laws for arbitrary vectors so that

$$\beta_p\beta_q\beta_r+\beta_r\beta_q\beta_p=(\mathbf{p}\cdot\mathbf{q}/pq)\beta_r+(\mathbf{r}\cdot\mathbf{q}/rq)\beta_p,$$

where $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are space vectors. In particular

$$\beta_p\beta_q\beta_p=(\mathbf{p}\cdot\mathbf{q}/pq)\beta_p,$$

$$\beta_p{}^2\beta_q+\beta_q\beta_p{}^2=(\mathbf{p}\cdot\mathbf{q}/pq)\beta_p+\beta_q.$$

Useful relations involving β_4 are:

$$\beta_4\beta_p\beta_4=0,\quad \beta_p\beta_4\beta_p=0,$$

$$\beta_p\beta_4{}^2+\beta_4{}^2\beta_p=\beta_p,\quad \beta_4\beta_p{}^2+\beta_p{}^2\beta_4=\beta_4.$$

Thus

$$\beta_4H_p\beta_4=\mu c^2\beta_4,$$

$$\beta_4H_p{}^2\beta_4=\mu c^2\beta_4{}^2+p^2\beta_4{}^2\beta_p{}^2\beta_4{}^2.$$

Because of the complexity of the differential cross section, we confine ourselves in this paper to energies such that n, k, $l\gg\mu c^2$. The first nonvanishing terms in P, Q, R are proportional to $(\mu c^2)^2$ and consist of two types:

$$\beta_e\beta_a\beta_4\beta_l{}^2\beta_4\beta_a\beta_e\beta_4\beta_k{}^2\beta_4$$

and

$$\beta_k\beta_e\beta_4\beta_a{}^2\beta_4\beta_l{}^2\beta_4\beta_a{}^2\beta_4\beta_e\beta_k\beta_4{}^2. \tag{18}$$

No simple general methods of evaluating spurs of such expressions have been found, the introduction of Kemmer's η matrices often being complicated and difficult to handle without dealing with the separate components, since these η are not vectors. Kemmer's treatment shows immediately that any product containing an odd number of β_4 or β_p will have a vanishing spur.

It was found useful to introduce a new pseudovector matrix γ, with components $\gamma_1=i\beta_2\beta_4\beta_3$, $\gamma_2=i\beta_3\beta_4\beta_1$, $\gamma_3=i\beta_1\beta_4\beta_2$ which are Hermitian and obey the same commutation rules as the β_k. The principal advantage of this substitution lies in the reduction of the number of matrices to be considered. In addition, the γ_i are actually simpler in form than the β_i which leads to further simplifying properties such as $\gamma_p\beta_q\gamma_r=0$, and $\gamma_i\gamma_j\gamma_k=0$ for j perpendicular to i and k. Further, we can write

$$i\beta_p\beta_4\beta_q=[\mathbf{p}\times\mathbf{q}]_k\gamma_k/pq=r\gamma_r/pq,$$

where $\mathbf{r}=\mathbf{p}\times\mathbf{q}$ and $\gamma_r=r_k\gamma_k/r$.

The product of an odd number of γ has a vanishing spur since it contains an odd number of β_4. It is easily shown that Spur $\gamma_i{}^2=2$, and Spur $\gamma_p\gamma_q=2\mathbf{p}\cdot\mathbf{q}/pq$. To evaluate Spur $\gamma_p\gamma_q\gamma_r\gamma_s$ which is a general form of the spur of the first of the expressions (18) which we wish to evaluate, we proceed as follows:

$$\mathrm{Spur}\ \gamma_p\gamma_q\gamma_r\gamma_s=\mathrm{Spur}\ \gamma_s\gamma_p\gamma_q\gamma_r=\mathrm{Spur}\ \gamma_r\gamma_q\gamma_p\gamma_s,$$

since it is real, then

$$\mathrm{Spur}\ \gamma_p\gamma_q\gamma_r\gamma_s=\tfrac{1}{2}\,\mathrm{Spur}\ (\gamma_p\gamma_q\gamma_r\gamma_s+\gamma_r\gamma_q\gamma_p\gamma_s)$$

$$=\tfrac{1}{2}\,\mathrm{Spur}\ [(\mathbf{p}\cdot\mathbf{q}/pq)\gamma_r+(\mathbf{r}\cdot\mathbf{q}/rq)\gamma_p]\gamma_s$$

by using the commutation rules, and finally

$$\text{Spur}\ \gamma_p\gamma_q\gamma_r\gamma_s=\frac{(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{s})+(\mathbf{p}\cdot\mathbf{s})(\mathbf{r}\cdot\mathbf{q})}{pqrs}.$$

A useful commutation rule connecting the β and γ is

$$\gamma_p\beta_q\beta_r+\beta_r\beta_q\gamma_p=2((\mathbf{q}\cdot\mathbf{r})/qr)\gamma_p-((\mathbf{p}\cdot\mathbf{r})/pr)\gamma_q.$$

This gives immediately

$$\begin{aligned}\text{Spur}\ \gamma_p\gamma_q\beta_r\beta_s&=-\text{Spur}\ \gamma_p\beta_s\beta_r\gamma_q+2(\mathbf{r}\cdot\mathbf{s}/rs)\ \text{Spur}\ \gamma_p\gamma_q-(\mathbf{q}\cdot\mathbf{s}/qs)\ \text{Spur}\ \gamma_p\gamma_r\\&=\frac{2(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{s})-(\mathbf{p}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{s})}{pqrs}.\end{aligned}$$

The spur of the second of the expressions (18) can now be treated by noting that

$$\beta_4\beta_a{}^2\beta_4\beta_l{}^2\beta_4\beta_a{}^2\beta_4=\beta_l{}^2\beta_4\beta_a{}^2\beta_4,$$

since $\beta_a{}^2\beta_l{}^2=\beta_l{}^2\beta_a{}^2$. The resulting expression

$$\beta_4\beta_k\beta_e\beta_l{}^2\beta_4\beta_a{}^2\beta_4\beta_e\beta_k\beta_4$$

can be written

$$-\beta_l\beta_k\beta_4\beta_l{}^2\beta_4\beta_a{}^2\beta_4\beta_l\beta_k\beta_4+(\mathbf{e}\cdot\mathbf{k}/k)\beta_4\beta_l{}^2\beta_4\beta_a{}^2\beta_4\beta_e\beta_k\beta_4.$$

The first is of the general form $\gamma_p\gamma_q\gamma_r\gamma_s$ and the second of the form $\gamma_p\gamma_q\beta_r\beta_s$; the spurs of both these expressions have been evaluated. We thus obtain another general formula

$$\text{Spur}\ \beta_p\beta_q\gamma_r\gamma_s\gamma_t\gamma_u=\frac{(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{s})(\mathbf{t}\cdot\mathbf{u})+(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t})-(\mathbf{p}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t})}{pqrstu}.$$

In concluding this section, we give the spurs of a few simple products of β which enter into problems such as the scattering of mesotrons in a Coulomb field or the scattering of light by mesotrons. They are evaluated by methods similar to those given above.

Thus

$$\beta_p\beta_q\beta_4{}^2=-\beta_4\beta_q\beta_p\beta_4+(\mathbf{p}\cdot\mathbf{q}/pq)\beta_4{}^2.$$

$$\text{Spur}\ \beta_p\beta_q\beta_4{}^2=\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{q}/pq)\ \text{Spur}\ \beta_4{}^2=3\mathbf{p}\cdot\mathbf{q}/pq.$$

Also

$$\beta_p\beta_q\beta_4{}^2\beta_r\beta_s=-\beta_p\beta_q\beta_4\beta_s\beta_r\beta_4+(\mathbf{r}\cdot\mathbf{s})/rs)\beta_p\beta_q\beta_4{}^2,$$

$$\begin{aligned}\text{Spur}\ \beta_p\beta_q\beta_4{}^2\beta_r\beta_s&=\frac{2(\mathbf{q}\times\mathbf{s})\cdot(\mathbf{r}\times\mathbf{p})}{pqrs}+\frac{3(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{s})}{pqrs}\\&=\frac{2(\mathbf{q}\cdot\mathbf{r})(\mathbf{p}\cdot\mathbf{s})+(\mathbf{p}\cdot\mathbf{q})(\mathbf{r}\cdot\mathbf{s})}{pqrs}.\end{aligned}$$

III

In this section, we indicate the manner in which the integration of the differential cross section was carried out. After averaging over the polarizations of the incident γ-ray, the cross section appears in terms of k, θ_+, φ_+ of the positive mesotron and l, θ_-, φ_- of the negative mesotron, referred to the direction of the γ-ray as the polar axis. In place of φ_+ and φ_-, the azimuthal angles, we introduce $(\varphi_+-\varphi_-)$ and $\frac{1}{2}(\varphi_++\varphi_-)$: Only the former has physical significance, integration over the latter yields a factor 2π.

We then introduce $x=1-\cos\theta_+$, $y=1+\cos\theta_-$, $z=\cos(\varphi_+-\varphi_-)$, so that $0\leqslant x\leqslant 2$, $0\leqslant y\leqslant 2$ and we can limit z by $-1\leqslant z\leqslant 1$ by introducing a factor 2. The cross section can then be written

$$d\varphi=\pi\alpha\frac{kldkdxdydz}{2n(\mu c^2)^4(1-z^2)^{\frac{1}{2}}}\frac{(V(q))^2}{(2\pi\hbar c)^4}\{A+bzB+b^2z^2C+b^3z^3D\},\tag{19}$$

where

$$\begin{aligned}A=\tfrac{1}{4}\{&[(6k^4+8k^3l+22k^2l^2)x-(3k^4+4k^3l+11k^2l^2)x^2]\\&+[(22k^2l^2+8kl^3+6l^4)+(4k^3l-32k^2l^2+4kl^3)x+(-2k^3l+14k^2l^2)]y\\&+[-(11k^2l^2+4kl^3+3l^4)+(14k^2l^2-2kl^3)x-6k^2l^2x^2]y^2\},\end{aligned}$$

$$B=-\tfrac{1}{2}\{[(3k^2-4kl+3l^2)+(-k^2+5kl)x]+[(5kl-l^2)-4klx]y\},\qquad q^2=2(a-bz),$$

$$C=-\tfrac{1}{2}\{2/y+2/x-1\},\qquad a=k^2x+l^2y+klxy,$$

$$D=-1/klxy,\qquad b=kl\sin\theta_+\sin\theta_-.$$

For pure Coulomb field, $V(q)=4\pi Ze^2\hbar^2c^2/q^2$ whereas for a field $(Ze/r)(1-e^{-r/d})$

$$V(q)=\frac{4\pi Ze^2\hbar^2c^2}{q^2}\frac{(\hbar c/d)}{q^2+(\hbar c/d)^2}.$$

The first integral over z is simple. The integrals over y and x can be shortened by using the symmetry in y and x, but even then they are tedious though elementary. There results for a Coulomb field

$$d\sigma=\frac{\alpha Z^2e^4}{2(\mu c^2)^4}\frac{kldk}{n}\left\{-\frac{1}{3}+\frac{9k^2+9kl+8l^2}{12kn}\ln\frac{n^2}{l^2}+\frac{8k^2+9kl+9l^2}{12ln}\ln\frac{n^2}{k^2}\right\} \quad (20)$$

and for the cut-off field

$$d\sigma=\frac{\pi\alpha Z^2e^4\hbar cdk}{48(\mu c^2)^4dn^2}\{7k^2+12kl+7l^2\}. \quad (21)$$

The differential cross section for bremsstrahlung, where a mesotron of momentum $\mathbf{k}$ is scattered in the electric field of a nucleus, emitting a quantum $\mathbf{n}$ and leaving the mesotron with momentum $\mathbf{l}$, can be obtained from that for pair creation by redefinition of the variables x, y, z, changing the sign of l in the square of the matrix element, and introducing different expressions for the density of states and average over the initial states. Thus the angle integrals are unchanged and the cross sections for bremsstrahlung are simply obtained from (20) and (21) by changing the sign of l in the brackets and changing the density factors outside the brackets. In the Coulomb field,

$$d\sigma=\frac{\alpha Z^2e^4nldn}{3(\mu c^2)^4k}\left\{-\frac{1}{3}+\frac{9k^2-9kl+8l^2}{12kn}\ln\frac{n^2}{l^2}-\frac{8k^2-9kl+9l^2}{12ln}\ln\frac{n^2}{k^2}\right\} \quad (22)$$

and for the cut-off field

$$d\sigma=\frac{\pi\alpha Z^2e^4\hbar cdn}{72(\mu c^2)^4k^2d}\{7k^2-12kl+7l^2\}, \quad (23)$$

where now $n=k-l$.

These integrate to the cross sections given in Section I with E equal to n for pair creation and to k for bremsstrahlung.

IV

We now apply the method of virtual quanta to obtain approximately the radiation emitted by a very fast mesotron in the neighborhood of a nucleus. The method can also be applied to the calculation of pair production, but since only the bremsstrahlung is directly connected with cosmic-ray bursts, we will restrict ourselves to just this effect.

Let the mesotron move with velocity v and energy E past a nucleus of charge Ze. Let us consider the process in the Lorentz frame where the mesotron is initially at rest and is scattered by the nucleus which, in this frame, is passing by the mesotron with velocity v. For $v\sim c$, the Coulomb field of the nucleus is greatly contracted longitudinally so that, at not too small distances, the field is largely transverse and can, by Fourier analysis, be represented by light quanta traveling parallel to the direction of motion of the nucleus: The number N_{k_0} of these quanta per unit area of energy k_0 per unit energy at a distance r from the path is approximately

$$\begin{aligned} N_{k_0}&=\alpha Z^2/\pi^2r^2k_0 && \text{for } k_0<E\hbar/r\mu c,\\ N_{k_0}&=0 && \text{for } k_0>E\hbar/r\mu c. \end{aligned} \quad (24)$$

The lower limit of r is taken to be approximately the size of the larger of the two interacting particles; here it is the nucleus with $r_{\min}\sim d$. The cross section for the scattering of a photon of energy k_0 to one of energy k through an angle θ by a mesotron initially at rest has been calculated by Booth and Wilson[3] and is

$$\begin{aligned}\sigma(k_0,k,\theta)&=\tfrac{1}{2}(e^2/\mu c^2)^2d\Omega(k^2/k_0^2)\{(1+\cos^2\theta)\\ &\quad+(1/48(\mu c^2)^2)[kk_0(28-64\cos\theta+12\cos^2\theta)\\ &\quad+(k^2+k_0^2)(29-16\cos\theta+\cos^2\theta)]\}.\end{aligned} \quad (25)$$

From the scattered quanta we must pick out such k and θ that correspond to a definite energy ϵE, $0\leqslant\epsilon\leqslant1-(\mu c^2/E)$, in the nuclear rest system. These are determined for $v\sim c$ by $k\sim k_0(1-\epsilon)$,

$$1-\cos\theta=\epsilon\mu c^2/[k_0(1-\epsilon)],$$
$$d\Omega=2\pi\mu c^2d\epsilon/[k_0(1-\epsilon)^2],$$

$k_0 \geqslant \epsilon\mu c^2/[2(1-\epsilon)]$. The cross section for bremsstrahlung $\sigma(\epsilon, E)$ is then given by

$$\sigma(\epsilon, E)d\epsilon = \int_{\epsilon\mu c^2/[2(1-\epsilon)]}^{k_{0\,\max}} dk_0 \int_{fd}^{r_{\max}} 2\pi r dr \times N_{k_0}\sigma(k_0, k(\epsilon, k_0), \theta(\epsilon, k_0)).$$

The uncertainties in both the lower and upper limits of the r integral are here represented by f which is of order unity. If k_0 is written in units of μc^2 and r in units of $\hbar/\mu c$, we have

$$\sigma(\epsilon, E)d\epsilon = \alpha(e^2/\mu c^2)^2 Z^2 d\epsilon \int_{\epsilon/[2(1-\epsilon)]}^{k_{0\,\max}} dk_0 \int_{(5/6)fZ^{\frac{1}{3}}}^{E/\mu c^2 k_0} \times \frac{dr}{r}\left\{\frac{(2-2\epsilon+7\epsilon^2)}{12} + \frac{\epsilon(34-34\epsilon+7\epsilon^2)}{12(1-\epsilon)k_0} + \frac{96(1-\epsilon)^2+\epsilon^2(14-14\epsilon+\epsilon^2)}{24(1-\epsilon)^2 k_0^2} - \frac{4\epsilon}{(1-\epsilon)k_0^3} + \frac{2\epsilon^2}{(1-\epsilon)^2 k_0^4}\right\}. \quad (26)$$

Interchanging the order of integrations and integrating the above, we see the dependence of the various terms on the impact parameter r. These dependences are $1/r$, $\ln^2 r$, and $\ln r$ for the first, second and last three terms, respectively.

In the cut-off electromagnetic field of the nucleus, the neglect of the transverse momentum transfers (and the longitudinal components of the nuclear field) with respect to the longitudinal momentum transfers is justified in the zero momentum system where the mesotron energy is $(\frac{1}{2}\mu c^2 E)^{\frac{1}{2}}$ or in the mesotron rest system, but not in the nuclear rest system. However, the approximation is critical since the first term of (26) is proportional to the highest frequency available and inversely proportional to the smallest impact parameter, so that the ratio of the uncertainties in the upper and lower limits of r as written in (26) enters as a multiplicative constant in the first term, which at sufficiently high energies is dominant. In the other terms, it is the logarithm of the ratio of the upper and lower impact parameters which is an unessential approximation always present in calculations by this method. There is a difference in nature between the two approximations in the limits of the impact parameter r. The upper limit is approximate only as a mathematical convenience in the representation of N_{k_0}. On the other hand, the lower limit d of r is, to a certain extent, conceptually indeterminate since it supplies a cut-off in the space integral which is independent of the shape of the electromagnetic field near d; whereas the actual process depends on the Fourier analysis of this field. These two inaccuracies in the method of virtual quanta could, in principle, be removed by treating initially a suitable model for the field of a moving nucleus which would become constant for $r<d$, and by eliminating the mathematical approximations. This would, however, essentially reduce the calculation to the one we have performed in the previous section, though in a different Lorentz system.

It is for these reasons that the first term in the cross section was calculated by the better approximations of Section III. The factor f which expresses the ratio of uncertainties in the upper and lower limits of r was then taken to be $6/\pi$ to give agreement with the result of Section III. This does not, of course, guarantee that no further uncertainty exists in the logarithmic terms, but since it will be unimportant, it is convenient to omit it. The integration over the impact parameter in (26) then yields the factor

$$\ln \frac{\pi E}{5\mu c^2 Z^{\frac{1}{3}} k_0}.$$

The upper limit of the frequency integral will in general be given by $k_{0\,\max} = \pi E/5\mu c^2 Z^{\frac{1}{3}}$ where the logarithm vanishes. This gives a cross section

$$\sigma = \alpha\left(\frac{e^2}{\mu c^2}\right)^2 Z^2 d\epsilon\left\{\frac{\pi E}{5\mu c^2 Z^{\frac{1}{3}}}\left(\frac{2-2\epsilon+7\epsilon^2}{12}\right) + \frac{\epsilon(34-34\epsilon+7\epsilon^2)}{24(1-\epsilon)} \ln^2 \frac{2\pi E(1-\epsilon)}{5\mu c^2 Z^{\frac{1}{3}}\epsilon} + \left[\frac{16(1-\epsilon)}{3\epsilon} + \frac{13\epsilon}{12} - \frac{5\epsilon^3}{24(1-\epsilon)}\right] \ln \frac{2\pi E(1-\epsilon)}{5\mu c^2 Z^{\frac{1}{3}}\epsilon} - \frac{\epsilon(10-10\epsilon+3\epsilon^2)}{8(1-\epsilon)} - \frac{52}{9}\frac{(1-\epsilon)}{\epsilon}\right\}. \quad (27)$$

For very high energies the first two terms which depend on high frequencies and large couplings cannot be right. But neglecting possible reduction of the cross section by nonlinearities in the theory,[7] a lower limit to σ can be assigned by cutting off the frequency integral at $k_{0\,\max}=A$, some constant less than $2\hbar c/e^2$. There results

$$\sigma=\alpha\left(\frac{e^2}{\mu c^2}\right)^2 Z^2 d\epsilon\left\{\left(A+A\ln\frac{\pi E}{5A\mu c^2Z^{\frac{1}{3}}}\right)\right.$$
$$\times\frac{(2-2\epsilon+7\epsilon^2)}{12}+\frac{\epsilon(34-34\epsilon+7\epsilon^2)}{24(1-\epsilon)}$$
$$\times\left[\ln^2\frac{2\pi E(1-\epsilon)}{5\mu c^2Z^{\frac{1}{3}}\epsilon}-\ln^2\frac{\pi E}{5A\mu c^2Z^{\frac{1}{3}}}\right]$$

[7] A rigorous treatment might show that the presence of high Fourier components diminishes the contribution from the low frequencies. We are here ignoring this possibility. See J. R. Oppenheimer, Phys. Rev. **47**, 44 (1935).

$$+\left[\frac{16(1-\epsilon)}{3\epsilon}+\frac{13\epsilon}{12}-\frac{5\epsilon^3}{24(1-\epsilon)}\right]\ln\frac{2\pi E(1-\epsilon)}{5\mu c^2Z^{\frac{1}{3}}\epsilon}$$
$$\left.-\frac{\epsilon(10-10\epsilon+3\epsilon^2)}{8(1-\epsilon)}-\frac{52}{9}\frac{(1-\epsilon)}{\epsilon}\right\}\quad(28)$$

for $E>(5/\pi)\mu c^2Z^{\frac{1}{3}}A$. For $E<(5/\pi)\mu c^2Z^{\frac{1}{3}}A$, we get (27) above, as with no cut-off.

A consideration of cosmic-ray bursts based on these and other calculations is given in another paper.

In conclusion, the authors wish to express their appreciation to Professor J. R. Oppenheimer for continued advice and encouragement.

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Burst Production by Mesotrons

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Assuming that, under great absorbing thicknesses, cosmic-ray bursts are cascade showers from high energy soft secondaries produced in the shielding matter by mesotron-electron collisions and by mesotron bremsstrahlung, we have calculated the frequency of burst production as a function of burst size. For the mesotron of spin 1 and moment $e\hbar/2\mu c$, we have used the previously calculated knock-on formulae, supplemented by our own calculations of the bremsstrahlung; for the latter, the cross section has terms, significant for our work, in E, $\ln^2 E$, and $\ln E$. Up to energies close to 10^{11} ev, only slight modifications are introduced by omitting altogether those processes which cannot be treated by the Born approximation, and the *minimum* cross sections we used differ little from those given directly by the Born approximation. Using these cross sections, the cascade theory of showers, and a modified form of the Furry model to take into account the fluctuations, the frequency of burst production was calculated. The sea-level data of Schein and Gill give for the number of bursts of size greater than S, $N_S \backsim S^{-\gamma}$, with $\gamma=1.8$. Our calculations give for spin 1, $\gamma\sim 1.5$ and numerically too many by a factor of 20. Similar calculations for the mesotron of spin 0 give $\gamma\sim 1.8$ and the same in number as the observations within an uncertainty of about a factor 1.5. For spin $\frac{1}{2}$ and moment $e\hbar/2\mu c$, the bursts are approximately twice as numerous as for spin 0. This evidence thus favors spin 0, or possibly spin $\frac{1}{2}$, but tends to exclude spin 1.

I

COSMIC-RAY bursts, insofar as they involve high energies of order 10^9–10^{11} ev, provide a feasible test of relativistic mesotron theory. Experiments have shown that the ionization in bursts does not show the characteristic high initial recombination of that due to slow heavy particles. Furthermore, bursts frequently appear simultaneously in ionization chambers one of which is above the other, and sometimes are larger in the lower chamber.[1] This appears to be conclusive evidence that at least the majority of bursts are not due to several slow heavy particles resulting from a nuclear explosion or evaporation but, rather, are due to many fast electrons resulting from the cascade multiplication of a high energy soft ray in the material above the chamber. Now the transition curves of Nie, and Steinke and Schmidt[2] for bursts in lead show a maximum at $\sim$4 cm but no apparent decrease for thicknesses greater than 10 cm; bursts have also been observed at great depths underground.

[1] H. Nie, Zeits. f. Physik **99**, 776 (1936); H. Euler, Zeits. f. Physik **116**, 73 (1940).

[2] H. Nie, Zeits. f. Physik **99**, 453 (1936); E. C. Steinke and H. Schmidt, Zeits. f. Physik **115**, 740 (1940).